Available online at www.sciencedirect.com
SCIENCE DIRECT
International Journal of Heat and Mass Transfer 49 (2006) 2793–2801
International Journal of HEAT and MASS TRANSFER
www.elsevier.com/locate/ijhmt

Comparison of the solutions of a phase-lagging heat transport equation and damped wave equation with a heat source

Shengjun Su, Weizhong Dai *

Mathematics and Statistics, College of Engineering and Science, Louisiana Tech University, Ruston, LA 71272, USA

Available online 12 May 2006

Abstract

In this article, we study the difference between the solutions of the phase-lagging equation (PLE), a new heat conduction equation, and the damped wave equation (DWE) with a heat source. The exact solution of the PLE is obtained using the Laplace transform method, and is approximated using an approximate analytical method when the time lag is small since the exact solution is practically not computable for the small time lag case. Results show that the DWE is a good approximation to the PLE when the time lag is small.

1. Introduction

Heat transport at the microscale is very important in microtechnology applications. The lagging response must be included [1–6] under low temperature or high heat-flux conditions. Thus, the traditional Fourier's law [7]

$$\vec{q}(\vec{r},t) = -K\nabla u(\vec{r},t) \tag{1}$$

should be modified as follows [8]:

$$\vec{q}(\vec{r},t+\tau_0) = -K\nabla u(\vec{r},t), \tag{2}$$

where $\vec{q}$ is the heat flux vector, K is the thermal conductivity, u is the absolute temperature, $\vec{r}$ is the position vector, and t is the time. Here, $\tau_0(>0)$ represents the time lag required to establish steady thermal conduction in a volume element once a temperature gradient has been imposed across it. This lagging response describes the heat flux vector and the temperature gradient occurring at different instants of time in the heat transfer process. This quantity has been experimentally determined for a varieties of materials [1,9,10]. Combined with the energy conservation law

$$\rho C_p \frac{\partial u(\vec{r},t)}{\partial t} + \nabla\cdot\vec{q}(\vec{r},t) = Q, \tag{3}$$

where ρ is the mass density, C_p is the specific heat at constant pressure, and Q is the heat source, Eq. (2) results in the phase-lagging equation (PLE) as follows:

$$\frac{\partial u(\vec{r},t+\tau_0)}{\partial t} = \kappa\nabla^2 u(\vec{r},t) + Q, \tag{4}$$

where $\kappa = K/(\rho C_p)$ is the thermal diffusivity. On the other hand, approximating Eq. (4) by its first-order Taylor series expansion yields the damped wave equation (DWE) [11–24]

$$\frac{\partial u(\vec{r},t)}{\partial t} + \tau_0\frac{\partial^2 u(\vec{r},t)}{\partial t^2} = \kappa\nabla^2 u(\vec{r},t) + Q. \tag{5}$$

Recently, we have studied the difference between the solutions of the phase-lagging heat transport equation and the damped wave equation by investigating the solutions of a test problem [25]. The solutions of the phase-lagging heat transport equation were obtained using the Laplace transform method and an approximate analytical method [26].

In this study, we extend our study to the phase-lagging heat transport equation with a heat source and compare the difference between the solutions of the phase-lagging heat transport equation and the damped wave equation with a heat source by investigating the solutions of a test problem.

* Corresponding author. Tel.: +1 318 257 3301; fax: +1 318 257 2562.
E-mail address: dai@coes.latech.edu (W. Dai).

0017-9310/$ - see front matter © 2006 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijheatmasstransfer.2006.03.002

Nomenclature

C_i	coefficient in a series	$\bar{T}(s)$	Laplace transform of $T(t)$
K	thermal conductivity	$u(x,t)$	dimensionless temperature
κ	thermal diffusivity	β	π^2
M	integer	∇	gradient operator
Q	heat source	Δt, Δx	time increment and grid size, respectively
$\vec{q}$	heat flux	λ_0	time lag
$T(t)$	function of t	τ_0, τ_c	values of the dimensionless time lag

2. Problem formulation and solution

Consider a simple 1D phase-lagging heat transport equation with a time delay τ_0 and a heat source $Q = e^{-t}\sin[\pi x]$, coupled with initial and boundary conditions as follows:

$$\frac{\partial u(x,t+\tau_0)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + e^{-t}\sin[\pi x], \quad (x,t) \in (0,1)\times(0,\infty); \tag{6a}$$

$$u(0,t) = 0, u(1,t) = 0, \quad t > 0; \tag{6b}$$

$$u(x,0) = \sin[\pi x], \quad (x,t) \in (0,1)\times[-\tau_0, 0]. \tag{6c}$$

Using the first-order Taylor series expansion of Eq. (6a), we obtain the damping wave equation with initial and boundary conditions as follows:

$$\frac{\partial u(x,t)}{\partial t} + \tau_0 \frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2} + e^{-t}\sin[\pi x], \quad (x,t) \in (0,1)\times(0,\infty) \tag{7a}$$

$$u(0,t) = 0, u(1,t) = 0, \quad t > 0 \tag{7b}$$

$$u(x,0) = \sin[\pi x], \partial u(x,0)/\partial t = 0, \quad (x,t) \in (0,1)\times[-\tau_0, 0]. \tag{7c}$$

The exact solution of Eqs. (7a)–(7c) can be obtained using the separation of variables method as follows:

$$u(x,t) = \sin[\pi x]\cdot\begin{cases} e^{-t/2\tau_0}\left[\cosh(\omega t) + \frac{\sinh(\omega t)}{\sqrt{|\Delta|}}\right] + \frac{4\tau_0^2}{(1-2\tau_0)^2-\Delta}e^{-t} \\ -\frac{2\tau_0^2}{\sqrt{|\Delta|}(1-\sqrt{|\Delta|}-2\tau_0)}e^{\frac{-1+\sqrt{|\Delta|}}{2\tau_0}t} \\ +\frac{2\tau_0^2}{\sqrt{|\Delta|}(1+\sqrt{|\Delta|})}\frac{(-2\tau_0)}{e}^{\frac{-1-\sqrt{|\Delta|}}{2\tau_0}t}, \quad \tau_0 < \tau_c \\ e^{-t/2\tau_0}\left[1 + \frac{t}{2\tau_0} - (\frac{2\tau_0}{1-2\tau_0})^2 - \frac{2\tau_0}{1-2\tau_0}t\right] \\ +(\frac{2\tau_0}{1-2\tau_0})^2 e^{-t}, \quad \tau_0 = \tau_c \\ e^{-t/2\tau_0}\left[\cos(\omega t) + \frac{\sin(\omega t)}{\sqrt{|\Delta|}}\right] + e^{(2\alpha+1)t}\cdot\frac{1}{(\alpha+1)^2+\omega^2} \\ +e^{\alpha t}\cdot\frac{1}{(\alpha+1)^2+\omega^2}[\cos(\omega t) - \frac{\alpha+1}{\omega}\sin(\omega t)], \quad \tau_0 > \tau_c \end{cases} \tag{8}$$

where $\tau_c = 1/4\pi^2$ is a critical value of the thermal lag time, $\alpha = \frac{-t}{2\tau_0}$, $\omega = (2\tau_0)^{-1}\sqrt{|\Delta|}$, and $\Delta = 1 - 4\pi^2\tau_0$. Here, we reject the case $\tau_0 > \tau_c$, because it allows u to assume negative values. This can be seen that if the present Q is replaced by $Q_0 e^{-t}\sin[\pi x]$, where Q_0 is a non-negative constant, then $\tau_0 > \tau_c$ would have to rejected so that $u < 0$ could never occur when $Q_0 = 0$. Furthermore, it is noted that by letting $\tau_0 \to 0$ in Eq. (8), one may recover the classic Fourier-based solution:

$$u(x,t) = \sin[\pi x]\cdot\left(\frac{e^{-t}}{\pi^2-1} + \frac{\pi^2-2}{\pi^2-1}e^{-\pi^2 t}\right). \tag{9}$$

We now solve Eqs. (6a)–(6c) using the Laplace transform method. To this end, we first assume the solution to be $u(x,t) = T(t)\sin[\pi x]$ and substitute it into Eq. (6a). This gives

$$T'(t+\tau_0) + \pi^2 T(t) = e^{-t}. \tag{10}$$

That is,

$$T'(t) + \pi^2 T(t-\tau_0) = e^{-(t-\tau_0)}, \tag{11}$$

where $T(t) = 1$, when $t \in [-\tau_0, 0]$, obtained from the initial condition, Eq. (6c). Multiplying Eq. (11) by e^{-st} and integrating it with respect to t over the interval $(0,\infty)$, we obtain

$$\int_0^\infty T'(t)e^{-st}dt + \pi^2\int_0^\infty T(t-\tau_0)e^{-st}dt = \int_0^\infty e^{-(t-\tau_0)}e^{-st}dt = \frac{e^{\tau_0}}{s+1}. \tag{12}$$

It can be seen that the first integral on the left-hand-side of Eq. (12) equals to $sT(s) - 1$ and the second one can be expressed as

$$\begin{aligned}\pi^2\int_0^\infty T(t-\tau_0)e^{-st}dt &= \pi^2\int_{-\tau_0}^\infty T(u)e^{-s(u+\tau_0)}du \\ &= \pi^2\int_0^\infty T(u)e^{-s(u+\tau_0)}du \\ &\quad + \pi^2\int_{-\tau_0}^0 T(u)e^{-s(u+\tau_0)}du \\ &= \pi^2 e^{-s\tau_0}\bar{T}(s) + \frac{\pi^2(1-e^{-s\tau_0})}{s}.\end{aligned} \tag{13}$$

Thus, we obtain the Laplace transform $\bar{T}(s)$ as follows

$$\begin{aligned}\bar{T}(s) &= \frac{1}{s} - \frac{\pi^2}{s(s+\pi^2 e^{-s\tau_0})} + \frac{e^{\tau_0}}{(s+1)(s+\pi^2 e^{-s\tau_0})} \\ &= \frac{1}{s} - \sum_{n=0}^{\infty}(-1)^n \pi^{2(n+1)} e^{-s\tau_0 n} s^{-n-2} \\ &\quad + \frac{e^{\tau_0}}{s+1}\sum_{n=0}^{\infty}(-1)^n \pi^{2n} e^{-s\tau_0 n} s^{-n-1}. \qquad (14)\end{aligned}$$

Inverting Eq. (14) term-by-term using a table of inverses along with the properties of the Laplace transform and convolution theory [27], we obtain

$$\begin{aligned}T(t) &= 1 - \sum_{n=0}^{\lfloor t/\tau_0 \rfloor}(-1)^n \pi^{2(n+1)} \frac{(t-\tau_0 n)^{n+1}}{(n+1)!} \\ &\quad + e^{\tau_0}\sum_{n=0}^{\lfloor t/\tau_0 \rfloor}\left[\frac{(-1)^n \pi^{2n}}{n!}\int_0^t (\tau-\tau_0 n)^n e^{-(t-\tau)} d\tau\right], \qquad (15)\end{aligned}$$

where $\lfloor t/\tau_0 \rfloor$ stands for the largest integer that is less than or equal to t/τ_0. Hence, the solution of Eqs. (6a)–(6c) can be expressed as follows:

$$\begin{aligned}u(x,t) &= \left\{1 - \sum_{n=0}^{\lfloor t/\tau_0 \rfloor}(-1)^n \pi^{2(n+1)} \frac{(t-\tau_0 n)^{n+1}}{(n+1)!} \right. \\ &\quad \left. + e^{\tau_0}\sum_{n=0}^{\lfloor t/\tau_0 \rfloor}\frac{(-1)^n \pi^{2n} A(n)}{n!}\right\}\sin[\pi x] \\ &= \left\{\sum_{n=0}^{\lfloor t/\tau_0+1 \rfloor}(-1)^n \pi^{2n} \frac{(t-\tau_0 n-\tau_0)^n}{n!} \right. \\ &\quad \left. + e^{\tau_0}\sum_{n=0}^{\lfloor t/\tau_0 \rfloor}\frac{(-1)^n \pi^{2n} A(n)}{n!}\right\}\sin[\pi x], \qquad (16)\end{aligned}$$

where $A(n)$ is calculated by the formula

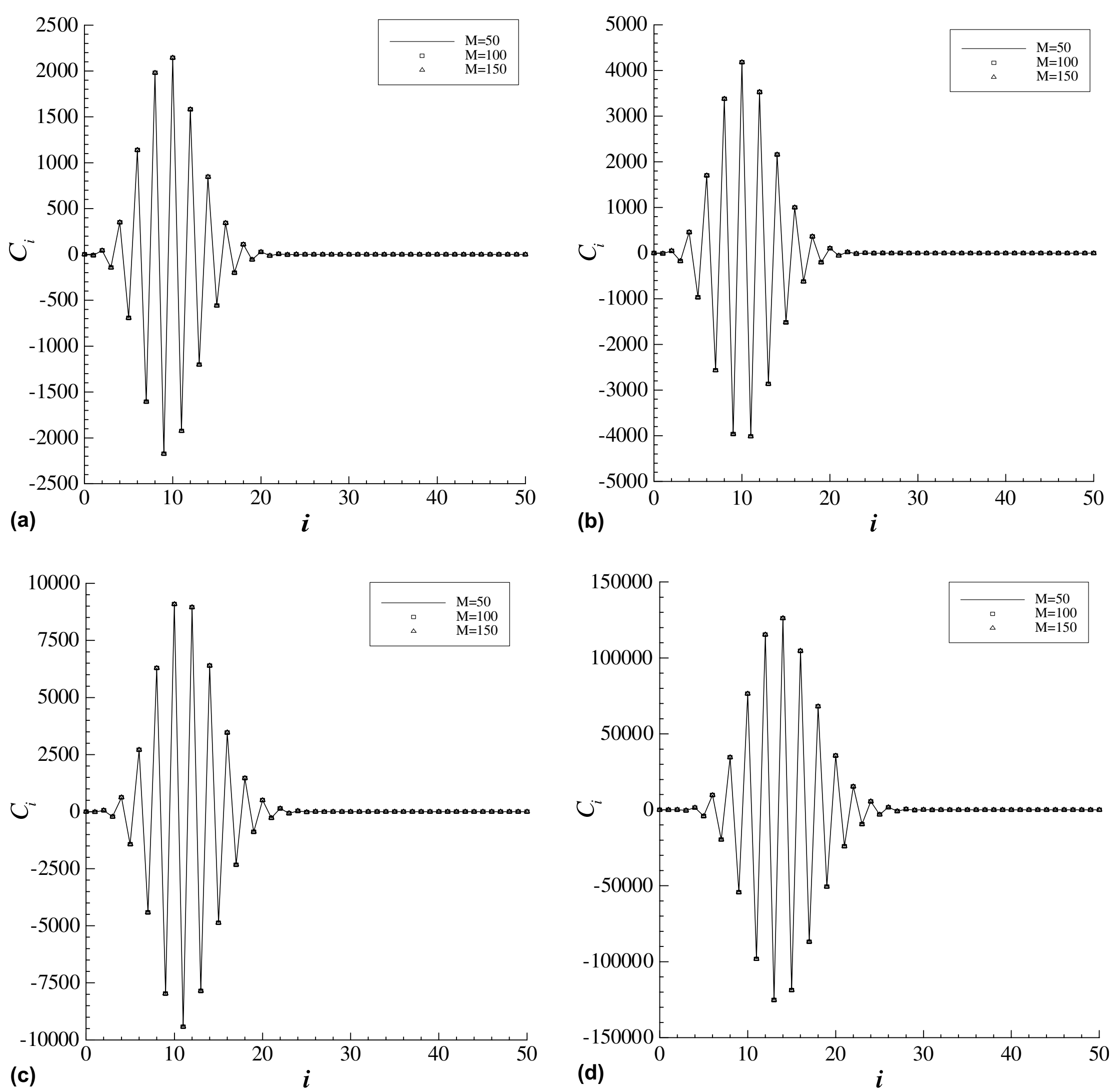

Fig. 1. Coefficient C_i for (a) $\tau_0 = 0$, (b) $\tau_0 = 0.25\tau_c$, (c) $\tau_0 = 0.5\tau_c$ and (d) $\tau_0 = \tau_c$.

$$
\begin{aligned}
A(n) &= \int_0^t (\tau-\tau_0 n)^n \mathrm{e}^{\tau-t}\mathrm{d}\tau \\
&= (\tau-\tau_0 n)^n \mathrm{e}^{\tau-t}\big|_0^t - n\int_0^t (\tau-\tau_0 n)^{n-1}\mathrm{e}^{\tau-t}\mathrm{d}\tau \\
&= [(t-\tau_0 n)^n - (-\tau_0 n)^n \mathrm{e}^{-t}] - n\Big[(\tau-\tau_0 n)^{n-1}\mathrm{e}^{\tau-t}\big|_0^t \\
&\quad -(n-1)\int_0^t (\tau-\tau_0 n)^{n-2}\mathrm{e}^{\tau-t}\mathrm{d}\tau\Big] \\
&= [(t-\tau_0 n)^n - (-\tau_0 n)^n \mathrm{e}^{-t}] - n\Big[(t-\tau_0 n)^{n-1} - (-\tau_0 n)^{n-1}\mathrm{e}^{-t}\Big] \\
&\quad + \cdots + (-1)^n n!(1-\mathrm{e}^{-t}) \\
&= (t-\tau_0 n)^n - (-\tau_0 n)^n \mathrm{e}^{-t} \\
&\quad + \sum_{m=1}^{n}\left\{(-1)^{n-m+1}\left(\prod_{j=m}^{n} j\right)\Big[(t-\tau_0 n)^{m-1} - (-\tau_0 n)^{m-1}\mathrm{e}^{-t}\Big]\right\}.
\end{aligned} \tag{17}
$$

It is noted that when $t \gg \tau_0$ that last term in $A(n)$ contains $n!$, which cancels the $n!$ in the denominator in $T(t)$ or $u(x,t)$. This implies that Eq. (16) is practically incomputable because π^{2n} in the numerator is sufficiently large. To overcome this difficulty, we introduce an approximate analytical method based on the idea in [25,26]. Letting

$$T(t) = \sum_{i=0}^{M} C_i t^i, \tag{18}$$

where M is a large integer and $C_0 = T(0)$, and substituting Eq. (18) into Eq. (10) give

$$\sum_{i=1}^{M} C_i \mathrm{i}(t+\tau_0)^{i-1} = -\pi^2 \sum_{i=0}^{M} C_i t^i + \mathrm{e}^{-t}. \tag{19}$$

Letting $t = 0$, we obtain

$$\sum_{i=1}^{M} C_i \mathrm{i}\tau_0^{i-1} = -\pi^2 C_0 + 1. \tag{20}$$

Differentiating Eq. (19) with respect to t and letting $t = 0$ give

$$\sum_{i=2}^{M} C_i \mathrm{i}(i-1)\tau_0^{i-2} = -\pi^2 C_1 - 1. \tag{21}$$

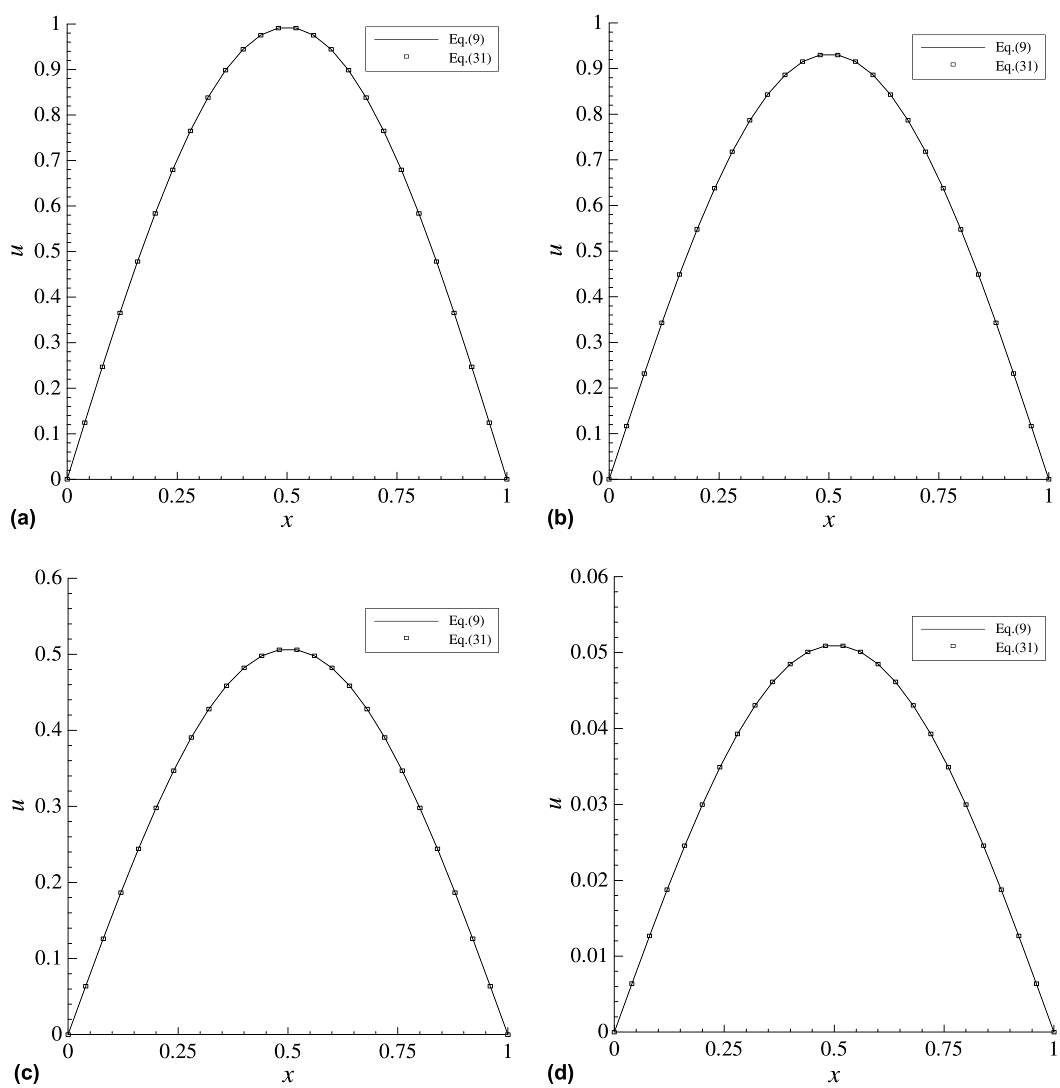

Fig. 2. u versus x for (a) $t = 2(\Delta t)$, (b) $t = 20(\Delta t)$, (c) $t = 200(\Delta t)$, (d) $t = 2000(\Delta t)$; $\Delta x = 0.04$; $\Delta t = 0.0004$; and $\tau_0 = 0$.

Differentiating Eq. (19) twice with respect to t and letting $t = 0$ give

$$\sum_{i=3}^{M} C_i i(i-1)(i-2)\tau_0^{i-3} = -2!\pi^2 C_2 + 1. \tag{22}$$

In general, we have

$$\sum_{i=k+1}^{M} C_i i(i-1)\cdots(i-k)\tau_0^{i-k-1} = -\pi^2 k! C_k + (-1)^k, \quad k = 0, 1, \ldots, M-1. \tag{23}$$

The next step is to solve the coefficients, C_k, $k = 1, \ldots, M$. Letting $k = M-1$, we obtain from Eq. (23)

$$C_M M! = -\pi^2 (M-1)! C_{M-1} + (-1)^{M-1}, \tag{24}$$

which yields

$$C_{M-1} = a_1 C_M + b_1,$$

where

$$a_1 = -\frac{M}{\pi^2}, \quad b_1 = \frac{(-1)^{M-1}}{\pi^2 (M-1)!}. \tag{25}$$

Letting $k = M-2$, we obtain from Eq. (23)

$$C_{M-2} = a_2 C_M + b_2, \tag{26}$$

where

$$a_2 = -\frac{1}{\pi^2}\left[\frac{M(M-1)\tau_0}{1!} + \frac{a_1(M-1)}{0!}\right],$$

$$b_2 = \frac{(-1)^{M-2}}{\pi^2 (M-2)!} - \frac{(M-1)b_1}{1!\pi^2}. \tag{27}$$

In general, we obtain

$$C_{M-k} = a_k C_M + b_k, \quad k = 1, 2, \ldots M; \tag{28}$$

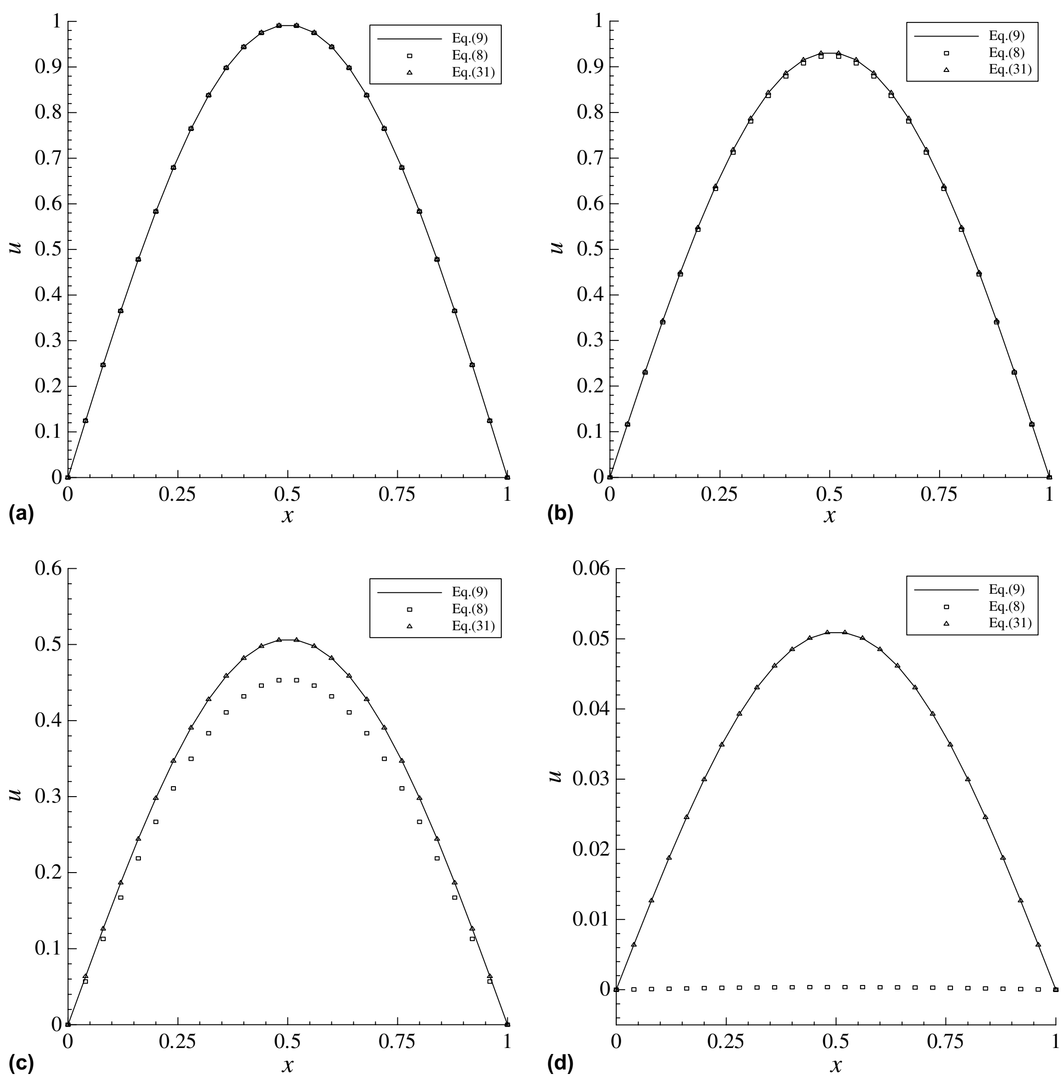

Fig. 3. u versus x for (a) $t = 2(\Delta t)$, (b) $t = 20(\Delta t)$, (c) $t = 200(\Delta t)$, (d) $t = 2000(\Delta t)$; $\Delta x = 0.04$; $\Delta t = 0.0004$; and $\tau_0 = 0.001\tau_c$.

where

$$a_k = \frac{1}{-\pi^2}\sum_{i=1}^{k}\left[\frac{a_{k-i}\tau_0^{i-1}}{(i-1)!}\cdot\prod_{j=M-k+1}^{M-k+i} j\right], \quad a_0 = 1; \tag{29a}$$

$$b_k = \frac{(-1)^{M-k}}{\pi^2(M-k)!} - \frac{1}{\pi^2}\sum_{i=1}^{k-1}\left[\frac{b_{k-i}\tau_0^{i-1}}{(i-1)!}\cdot\prod_{j=M-k+1}^{M-k+i} j\right]. \tag{29b}$$

Since $C_0 = T(0) = 1$, we can solve for C_M from Eq. (28) by letting $k = M$

$$C_M = (C_0 - b_M)/a_M. \tag{30}$$

Once C_M is determined, the rest of coefficients C_i $(i = 1, 2, \ldots, M)$ can be easily obtained from Eqs. (28) and (29) and hence an approximate analytical solution for the case of $t \gg \tau_0$ can be expressed as follows

$$u(x,t) = \sum_{i=0}^{M} C_i t^i \sin[\pi x]. \tag{31}$$

To determine how large the integer M should be, we have, in Fig. 1, plotted the coefficients C_i $(i = 0, 1, 2, \ldots, M)$, where $M = 50$, 100, 150, for $\tau_0 = 0, 0.25\tau_c$, $0.5\tau_c$, τ_c. It can be seen that for each value of τ_0 considered, the coefficients do not change significantly as the value of M is increased. Thus, we chose $M = 50$ in this study.

It should be pointed out that Tzou [8] has developed a numerical inversion algorithm for the Laplace transform. Briefly, if $\bar{T}(s) = L[T(t)]$, then

$$T(t) \approx \frac{e^{4.7}}{t}\left\{\frac{1}{2}\bar{T}\left(\frac{4.7}{t}\right) + \mathrm{Re}\left[\sum_{n=1}^{N}(-1)^n \bar{T}\left(\frac{4.7 + in\pi}{t}\right)\right]\right\},$$
$$t > 0, \tag{32}$$

where $N \gg 1$ is an integer and $\mathrm{Re}[\varsigma]$ denotes the real part of a complex number ς. However, $\bar{T}(s)$ in Eq. (16) is a series and must be calculated approximately.

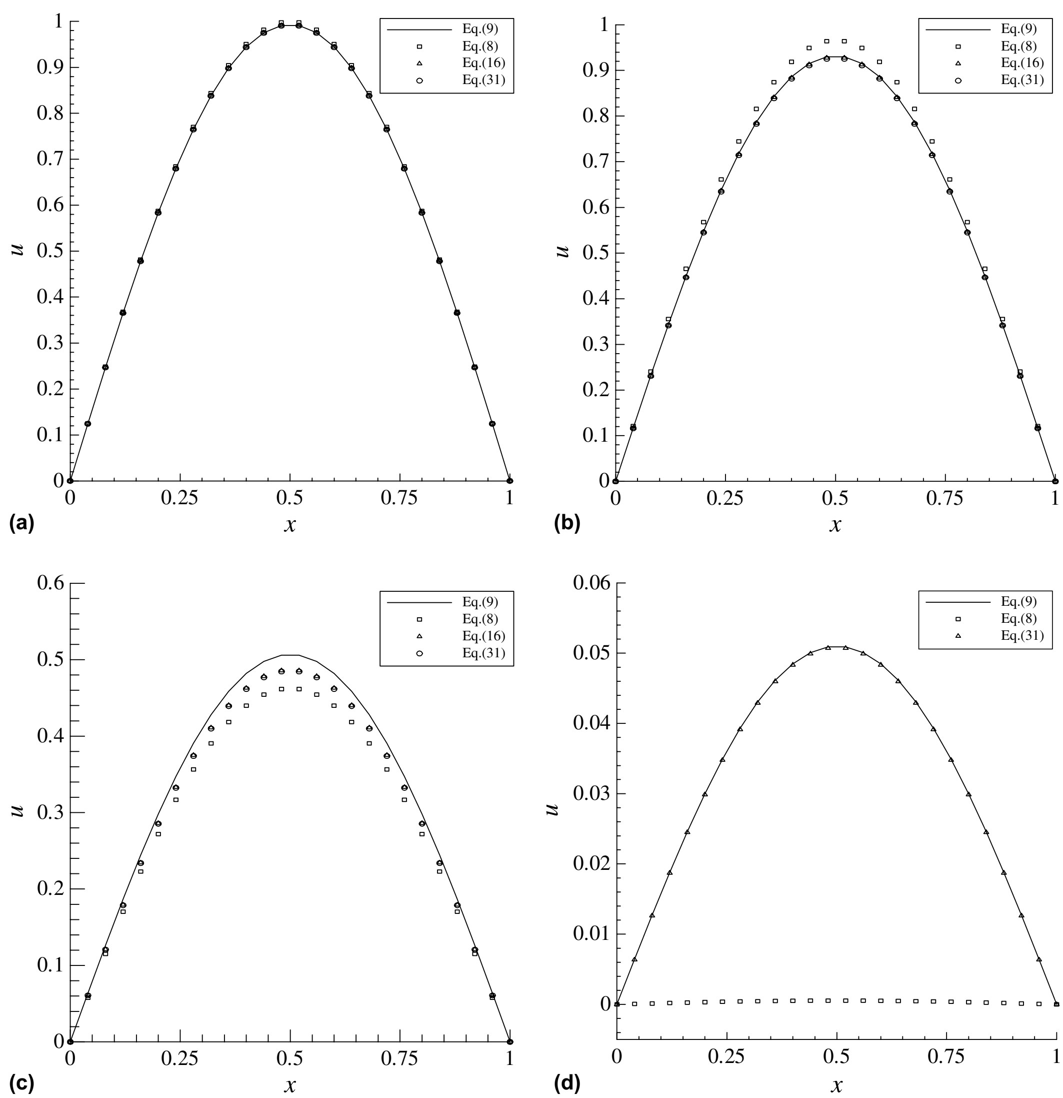

Fig. 4. u versus x for (a) $t = 2(\Delta t)$, (b) $t = 20(\Delta t)$, (c) $t = 200(\Delta t)$, (d) $t = 2000(\Delta t)$; $\Delta x = 0.04$; $\Delta t = 0.0004$; and $\tau_0 = 0.25\tau_c$.

3. Numerical results and testing

We have computed and plotted Eq. (8), the exact solution to IBVP (7) involving the DWE, the exact and approximate analytic solutions, Eq. (16) and Eq. (31), respectively, to IBVP (6) involving the phase-lagging equation, and for comparison Eq. (9), the exact solution of the traditional heat conduction equation. In our computation, we chose the time increment, Δt, and the grid size, Δx, to be 0.0004 and 0.04, respectively.

In Figs. 2–6, we have plotted the temporal evolution of the temperature, $u(x, t)$, versus x profile for $\tau_0 = 0$, $0.001\tau_c$, $0.25\tau_c$, $0.5\tau_c$, τ_c, and the time-sequence consisting of the times of $2(\Delta t)$, $20(\Delta t)$, $200(\Delta t)$, and $2000(\Delta t)$. Fig. 2 shows that these two solutions corresponding to Eqs. (9) and (31), are the same when $\tau_0 = 0$, as expected. The solution given in Eq. (16) is incomputable because t/τ_0 is undefined. Fig. 3 plots that the three solutions corresponding to Eqs. 8, 9 and 31 when $\tau_0 = 0.001\tau_c$. One may see that the solutions given in Eqs. (9) and (31) overlap and the level of the solution corresponding to Eq. (8) is lower when t is large. Again the solution given in Eq. (16) is incomputable since t/τ_0 is still large. When $\tau_0 = 0.25\tau_c$, one may see from Fig. 4 that the solutions corresponding to Eqs. (16) and (31) are very close to each other. When $t = 2000(\Delta t)$, the solution corresponding to Eq. (16) is not computable. Similar results can be seen in Figs. 5 and 6. Furthermore, one can see from Figs. 3–6 that by decreasing τ_0, the solutions given in Eq. (16) and (31) become "close" to the one corresponding to Eq. (8). This implies that when $\tau_0 \to 0$, the DWE is a good approximation to the phase-lagging heat transport equation.

4. Conclusion

The difference between the solutions of the PLE and the DWE with a heat source are compared by investigating the solutions of a test problem. The exact solution of

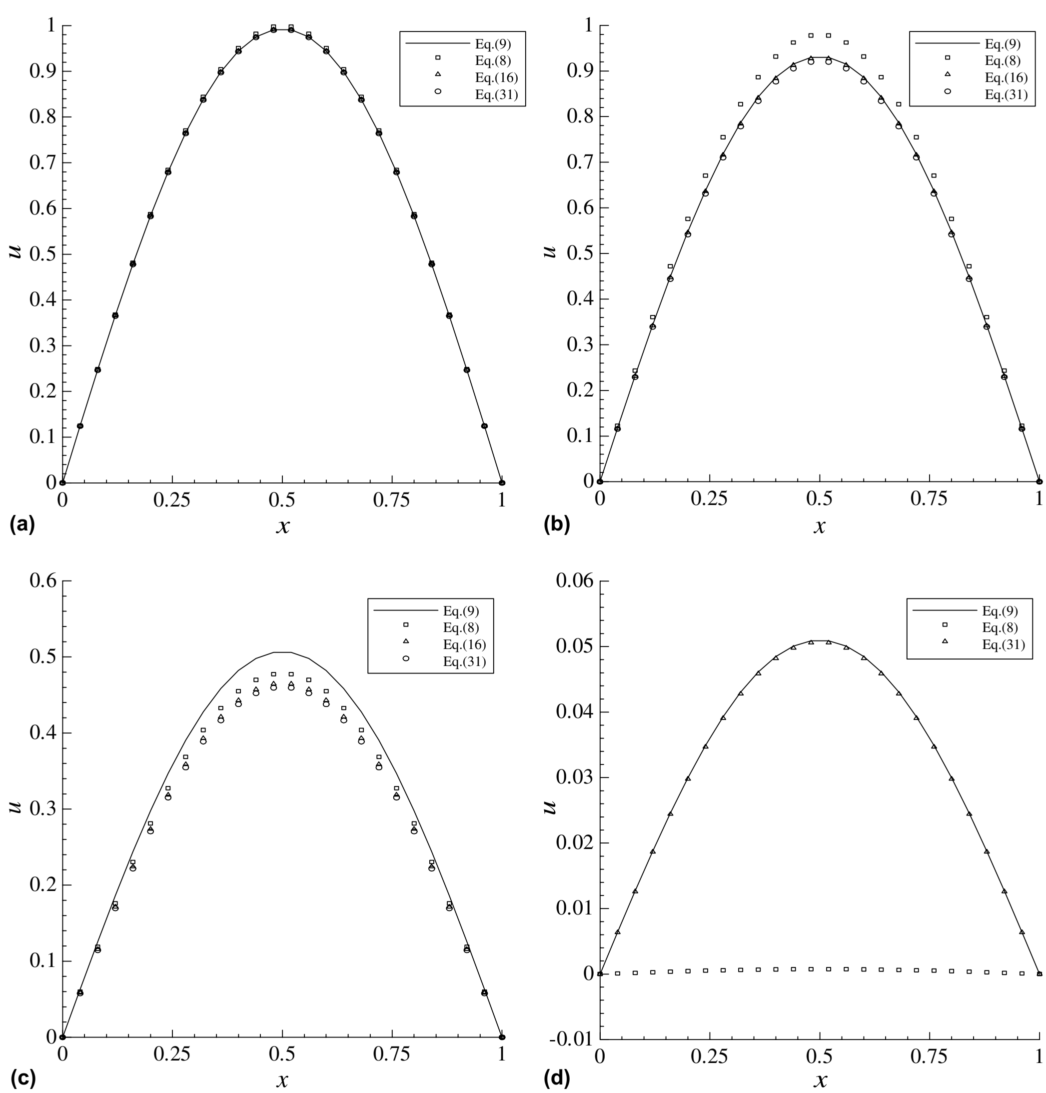

Fig. 5. u versus x for (a) $t = 2(\Delta t)$, (b) $t = 20(\Delta t)$, (c) $t = 200(\Delta t)$, (d) $t = 2000(\Delta t)$; $\Delta x = 0.04$; $\Delta t = 0.0004$; and $\tau_0 = 0.5\tau_c$.

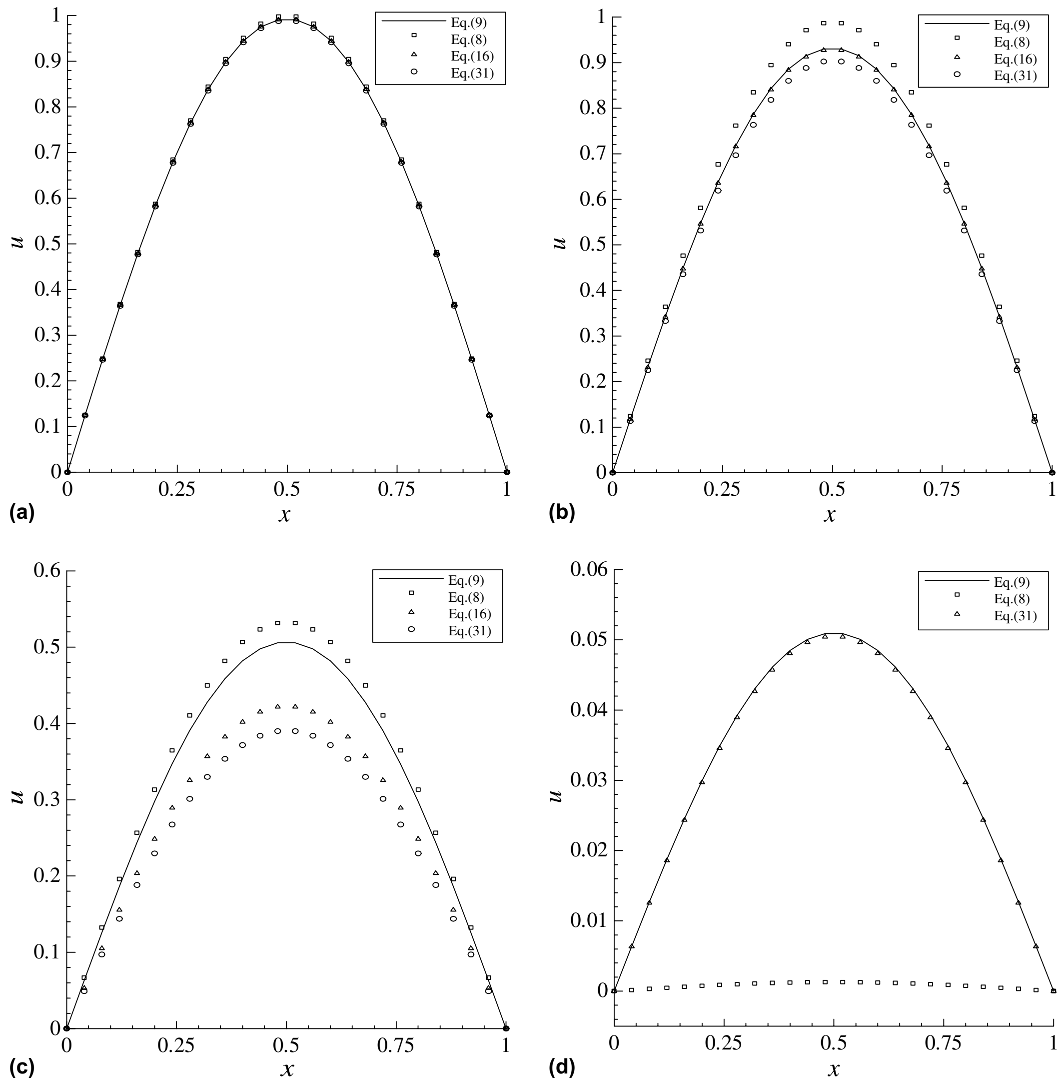

Fig. 6. u versus x for (a) $t = 2(\Delta t)$, (b) $t = 20(\Delta t)$, (c) $t = 200(\Delta t)$, (d) $t = 2000(\Delta t)$; $\Delta x = 0.04$; $\Delta t = 0.0004$; and $\tau_0 = \tau_c$.

the PLE is obtained using the Laplace transform method, and is approximated using an approximate analytic method when the time lag is small since the exact solution is practically not computable for the small time lag case. Results show that the DWE is a good approximation of the phase-lagging heat transport equation when τ_0 is small.

It should be pointed out that in the test problem, we chose a simple heat source $Q = e^{-t}\sin[\pi x]$. For a general heat source, one may expand the function into a Fourier sine series as described in [26].

Acknowledgements

This research was partially supported by a Louisiana Educational Quality Support Fund (LEQSF) grant. Contract no: LEQSF (2002-05)-RD-A-01. The authors thank the referees for their valuable suggestions.

References

[1] D.D. Joseph, L. Preziosi, Heat waves, Rev. Mod. Phys. 61 (1989) 41–73.
[2] A.A. Joshi, A. Majumdar, Transient ballistic and diffusive phonon heat transport in thin films, J. Appl. Phys. 74 (1993) 31–39.
[3] D.S. Chandrasekharaiah, Thermoelasticity with second sound: a review, Appl. Mech. Rev. 39 (1986) 355–376.
[4] D.S. Chandrasekharaiah, Hyperbolic thermoelasticity: a review of recent literature, Appl. Mech. Rev. 51 (1998) 705–929.
[5] W. Dreyer, H. Struchtrup, Heat pulse experiments revisited, Cont. Mech. Thermodyn. 5 (1993) 3–50.
[6] P.M. Jordan, P. Puri, Thermal stresses in a spherical shell under three thermoelastic models, J. Thermal Stress. 24 (2001) 47–70.
[7] M.N. Ozisik, Heat Conduction, 2nd ed., John Wiley & Sons, New York, 1993 (Chapters 1–2).
[8] D.Y. Tzou, Macro- to Microscale Heat Transfer: The Lagging Behavior, Taylor & Francis, Washington, DC, 1997 (Chapter 2).
[9] G. Caviglia, A. Morro, B. Straughan, Thermoelasticity at cryogenic temperatures, Int. J. Non-Linear Mech. 27 (1992) 251–263.
[10] P.J. Antaki, Hotter than you think, Mach. Des. 13 (1995) 116–118.

[11] M. Al-Nimr, M. Naji, On the phase-lag effect on the nonequilibrium entropy production, Microscale Thermophys. Eng. 4 (2000) 231–243.

[12] M. Al-Odat, M.A. Al-Nimr, M. Hamdan, Thermal stability of superconductors under the effect of a two-dimensional hyperbolic heat conduction model, Int. J. Numer. Meth. Heat Fluid Flow 12 (2001) 163–177.

[13] J.K. Chen, J.E. Beraun, Numerical study of ultrashort laser pulse interactions with metal films, Numer. Heat Transfer A 40 (2001) 1–20.

[14] J.K. Chen, J.E. Beraun, C.L. Tham, Investigation of thermal response caused by pulse laser heating, Numer. Heat Transfer A 44 (2003) 705–722.

[15] H.T. Chen, S.Y. Peng, P.C. Yang, Numerical method for hyperbolic inverse heat conduction problems, Int. Comm. Heat Mass Transfer 28 (2001) 847–856.

[16] I.H. Chowdhury, X. Xu, Heat transfer in femtosecond laser processing of metal, Numer Heat Transfer A 44 (2003) 219–232.

[17] O.J. Ilegbusi, A.U. Coskun, Y. Yener, Application of spectral methods to thermal analysis of nanoscale electronic devices, Numer. Heat Transfer A 41 (2002) 711–724.

[18] W.B. Lor, H.S. Chu, Propagation of thermal waves in a composite medium with interface thermal boundary resistance, Numer. Heat Transfer A 36 (1999) 681–697.

[19] W.B. Lor, H.S. Chu, Effect of interface thermal resistance on heat transfer in composite medium using the thermal wave model, Int. J. Heat Mass Transfer 43 (2000) 653–663.

[20] M.N. Ozisik, D.Y. Tzou, On the wave theory in heat conduction, Trans. ASME 116 (1994) 526–535.

[21] B. Pulvirenti, A. Barletta, E. Zanchini, Finite-difference solution of hyperbolic heat conduction with temperature-dependent properties, Numer. Heat Transfer A 34 (1998) 1040–7782.

[22] J.P. Wu, T.P. Shu, H.S. Chu, Transient heat-transfer phenomenon of two-dimensional hyperbolic heat conduction problem, Numer. Heat Transfer A 33 (1998) 635–654.

[23] J.P. Wu, H.S. Chu, Propagation and reflection of thermal waves in a rectangular plate, Numer. Heat Transfer A 36 (1999) 51–74.

[24] R.E. Mickens, P.M. Jordan, A positivity-preserving nonstandard finite difference scheme for the damped wave equation, Numer. Meth. Partial Differen. Equat. 20 (2004) 639–649.

[25] S. Su, W. Dai, P.M. Jordan, R.E. Mickens, Comparison of the solutions of a phase-lagging heat transport equation and damped wave equation, Int. J. Heat Mass Transfer 48 (2005) 2233–2241.

[26] W. Dai, R. Nassar, An approximate analytical method for solving 1D dual-phase-lagging heat transport equations, Int. J. Heat Mass Transfer 45 (2002) 1585–1593.

[27] D. Zwillinger, Handbook of Differential Equations, 3rd ed., Academic Press, 1998, pp. 231–237.